Linear Inequalities

Introduction

A relation of the type $f(x, y) > 0$, $f(x, y) < 0$, $f(x, y) \geq 0$, or $f(x, y) \leq 0$ is called an inequality or an inequation. It is also called a constraint or a condition. If $f(x, y)$ is linear in x and y, then the inequality is called a linear inequality or a linear inequation. For example

$$3x + 2y - 4 > 0, \qquad ax + by \leq c$$

are linear inequations in x and y. The solution set of a linear inequation is the set of all values of (x, y) which satisfy the given inequation.

The above examples involve two variables, namely x and y and the inequations are called linear inequations in two variables x and y.

The relations $ax + b \leq 0$ or $cx + d \geq 0$ are linear inequations in one variable x.

Solutions of linear inequations of one variable.

Let us suppose that a person has with him Rs. 200/- and he wishes to buy sugar which is available in 1 kg packets at a price of Rs. 19/- per packet. If x denotes the number of packets of sugar he buys, then the total amount spent is Rs, 19x. Since sugar can be bought in packets only, the entire amount of Rs. 200/- cannot be spent. Hence

$$19x < 200 \qquad \ldots(i)$$

and this statement is not an equation.

Clearly, x cannot be negative or a fraction. Now consider the inequation (i):

For $x = 0$, L.H.S. $= 19x = 0 < 200 =$ R.H.S. and this is true.

For $x = 1$, L.H.S. $= 19x = 19 < 200 =$ R.H.S which is true.

For $x = 2$, L.H.S. $= 19x = 38 < 200 =$ R.H.S. which is again true.

…….. ……. ……. ……. ……. ……. ……. ……

For $x = 9$, L.H.S. $= 19x = 19 \times 9 = 171 < 200 =$ R.H.S. This is true.

For $x = 10$, L.H.S. $= 19x = 19 \times 10 = 190 < 200 =$ R.H.S.,which is ture.

For $x = 11$, L.H.S. $= 19x = 209 < 200$, which is not true.

For all $x > 11$, the statement $19x < 200$ is not true.

We, thus, find that the only values of x which make the inequation a correct statement are $x = 0, 1, 2, \ldots, 10$. These values of x are called the solutions of the inequation (i).

Note:

(i) Equal numbers may be added to (or subtracted from) both sides of an inequality without affecting the sign of the inequality.

(ii) Both sides of an inequality can be multiplied (or divided) by the same positive number. But when both sides are multiplied or divided by a negative number, then the sign of the inequality is reversed.

Let us reconsider the inequation (i).

We have $19x < 200$

or $\frac{19x}{19} < \frac{200}{19}$

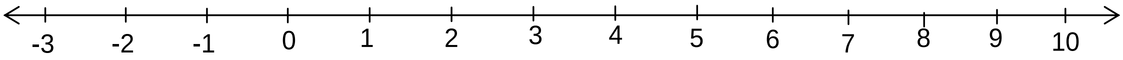
or $x < \frac{200}{19}$.

(i) When x is a natural number.

In this case the solutions are

1, 2, 3, ….10.

These solutions can be represented on the number line by ten points as shown



(ii) When x is an integer.

In this case, the solutions are

…., –3, –2, –1, 0, 1, 2, 3, 4, …., 10.

Illustration 1. Solve $3x + 5 < x - 7$ when

(i) x is an integer,

(ii) x is a real number.

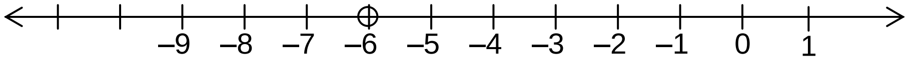
Solution: We have $3x + 5 < x - 7$ or, $3x + 5 - 5 < x - 7 - 5$

or, $3x - x < x - x - 12$ or, $\frac{2x}{2} < \frac{-12}{2}$

or $x < -6$.

(i) For integral solutions

$x = …., -9, -8, -7.$



(ii) For real values, the solutions are all real numbers x which are less than –6.

A circle over –6 implies that –6 is not included in the solution set.

Illustration 2. Solve $\frac{2x-3}{4} + 10 \geq 4 + \frac{4x}{3}$.

Solution: Here $\frac{2x-3}{4} + 10 \geq 4 + \frac{4x}{3}$ or, $12\left[\frac{2x-3}{4} + 10\right] \geq 12\left[4 + \frac{4x}{3}\right]$

or, $6x - 9 + 120 \geq 48 + 16x$ or, $6x + 111 \geq 48 + 16x$

or, $-10x \geq -63$ or, $\frac{-10x}{-10} \leq \frac{-63}{-10}$ or, $x \leq \frac{63}{10}$

i.e. all real numbers which are less than or equal to 6.3 are solutions of the given inequality.

Solutions of system of linear inequations in one variable

In order to solve a system of linear inequations, we first find the solution set of each of the inequations separately. Then, we find the values of the variable which are common to them or the intersection of all these sets.

Illustration 3. Solve for x, the following system of inequalities:

$$2x - 7 > 5 - x, \quad 11 - 5x \leq 1.$$

Solution: Here, $2x - 7 > 5 - x \Rightarrow 3x > 12$ or $x > 4$,

and $11 - 5x \leq 1 \Rightarrow 10 \leq 5x$ or $x \geq 2$.

From the above, we find that the value of x which satisfies both the inequalities

is $x > 4$.

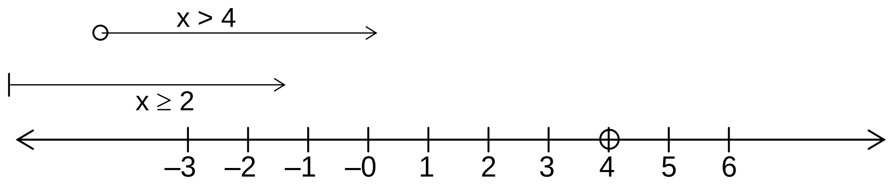


Illustration 4. Solve for x, the inequations $4x+3 \geq 2x+17, \quad 3x-5<-2$.

Solution: Here $4x + 3 \geq 2x + 17 \Rightarrow 2x \geq 14$ or $x \geq 7$.
Also, $3x – 5 \leq –2 \Rightarrow 3x < 3$ or $x < 1$.
Here x is less than one and at the same time greater than or equal to seven, which is not possible. Hence the given system has no solution.

Exercise 1:

Solve the inequations:

i) $4x + 3 > 3x; –(x + 3) + 4 \leq –2x + 5$

ii) $\frac{4x}{3}-\frac{9}{4}<x+\frac{3}{4}; \ \frac{7x-1}{3}-\frac{7x+2}{6}>x$

iii) $5x-7<3(x+3) \ ; \ 1-\frac{3x}{2} \leq x-4$

iv) $2(2x+3)-10<6(x-2); \ \frac{2x-3}{4}+6 \geq 2+\frac{4x}{3}$.

Linear inequations in two variables

A person has Rs. 200/- and wants to buy some pens and pencils. The cost of a pen is Rs. 16/- and that of a pencil is Rs. 6/-. If x denotes the number of pens and y, the number of pencils, then the total amount spent is Rs. (16 x + 6 y), and we must have

$16x + 6y \leq 200$. ...(i)

In this example x and y are whole numbers and can not be fractions or negative numbers. In this case we find the pair of values of x and y which make the statement (i) true. The set of such pairs is the solution set of (i).

To start with, let x = 0 so that

$6y \leq 200$ or $y \leq \frac{100}{3} = 33\frac{1}{3}$.

As such the values of y corresponding to x = 0, can be 0, 1, 2, 3, …., 33. Hence the solutions of (i) are (0, 0), (0, 1), (0, 2) ,….., (0, 33).

Similarly, the solutions corresponding to x = 1, 2, 3, …, 12 are

(1, 0), (1, 1), …. , (1, 30),

(2, 0), (2, 1), …., (2, 28),

…………………………..

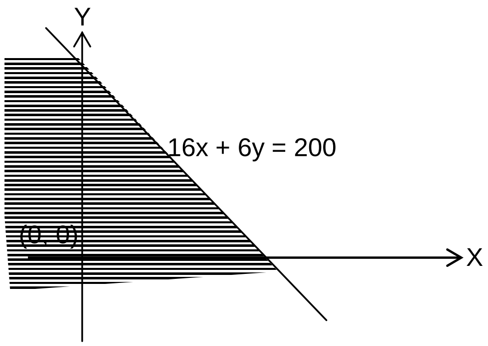
(11, 0), (11, 1), …, (11, 4),

(12, 0), (12, 1).

We note that the values of x and y cannot be more than 12 and 33 respectively. We also note that some of the pairs namely (5, 20), (8, 12), and (11, 4) satisfy the equation

$16x + 6y = 200$ which is a part of the given inequation.

Let us now extend the domain of x and y from whole number to the real numbers. Consider the equation 16 x + 6 y = 200
and then draw the straight line represented by it. This line divides the coordinate plane in two half planes.
For (0, 0), (i) yields, $0 \leq 200$ which is true and hence we conclude that (0, 0) belongs to the half plane represented by (i). Hence the solution of (i) consists of all the points belonging to the shaded half plane which consists of infinite number of points.



Note:
In order to identify the half plane represented by an inequation, it is sufficient to take any known point (not lying on the line) and check whether it satisfies the given inequation or not. If it satisfies, then the inequation represent that half plane, containing the known point, otherwise the inequation represents the other half plane.

For a set of linear inequations, in two variables, the solution region may be

(i) a closed region inside a polygon (bounded by straight lines),

(ii) an unbounded region (bounded partly by straight lines),

or (iii) empty.

Illustration 5. Draw the graph of the solution region satisfied by the inequalities $2x + y \leq 4$, $y \leq 2$, $x \geq 0$.

Solution: The solution region is bounded by the straight lines

$2x + y = 4$, ...(i)

y = 2, ...(ii)

x = 0. ...(iii)

The straight line (i) meets the coordinate axes at (2, 0) and (0, 4). Moreover, for (0, 0), the inequation $2x + y \leq 4$ gives $0 \leq 4$, which is true. Hence the half plane represented by $2x + y \leq 4$, contains the origin (0, 0).

Also for (0, 0), $y \leq 2$ gives $0 \leq 2$ which is true, so that $y \leq 2$ also contains the origin. With $x \geq 0$, the shaded region is the required solution region. All the boundary lines are part of the solution region.

Moreover, the solution region is an unbounded region, bounded by the three boundary lines.

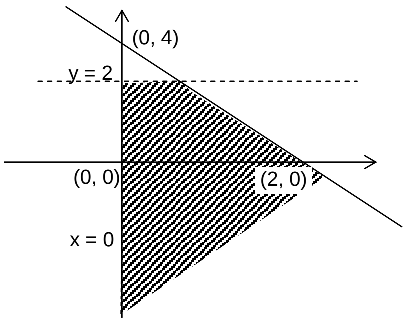
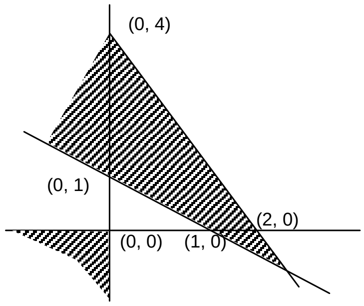


Illustration 6. Draw the graph of the solution region satisfied by the inequations $x + y \geq 1$, $2x + y \leq 4$, $x \leq 0$, $y \leq 0$.

Solution: The solution region is bounded by the straight lines

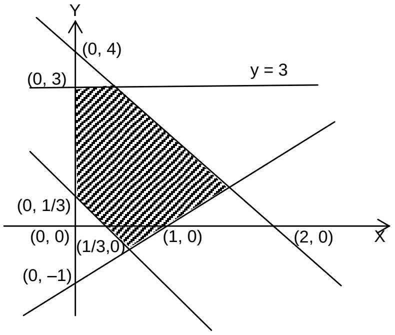


x + y = 1, (1)
2x + y = 4, (2)
x = 0, (3)
y = 0. (4)

The straight line (1) meets the coordinates axes at (1, 0) and (0, 1). For (0, 0),
x + y ≥ 1 gives 0 ≥ 1 which is not true.
Hence (0, 0) does not lie in the half plane represented by x + y ≥ 1. The line (2) meets the coordinate axes in (2, 0) and (0, 4) and 0 ≤ 4.Hence (0, 0) lies in the half plane 2x + y ≤ 4. Hence the region bounded by x + y ≥ 1 and 2x + y ≤ 4 belongs to either the fourth quadrant, or the first quadrant or the second quadrant. But the point belonging to x ≤ 0, y ≤ 0, all lie in the third quadrant.
Hence no points satisfies all the four given inequation.
Hence the solution set is empty.

Illustration 7. Find the solution region satisfied by the inequalities 2x + y ≤ 4, 3x + 3y ≥1, x – y ≤ 1, x ≥ 0, y ≤ 3.

Solution: The solution region is bounded by the straight lines



2x + y = 4, … (1)
3x + 3y = 1, … (2)
x – y = 1, … (3)
x = 0, … (4)
y = 3. … (5)

The points where the first three lines meet the x-axis are (2, 0), (1/3, 0), (1, 0).
The points where the first three lines meet the y-axis are (0, 4), (0, 1/3), (0, -1). Moreover, (0, 0) belongs to the half planes 2x + y ≤ 4 and x – y ≤ 1, y ≤ 3 but not to the half plane 3x + 3y ≥ 1.
The solution region is the shaded region. All the boundary lines are part of the solution region and the closed region inside a polygon.

Exercise 2.

Draw the graphs of the region satisfied by the solutions of

i) **2x + 3y ≥ 3,**
ii) **2x + y > 0, x + 2y ≤ 2,**
iii) **x – y ≥ 1, x + y ≤ 1, y > 1,**
iv) **2x + 3y ≤ 6, x + y ≥ 1, 2x – y – 4 ≤ 0, y ≥ 0.**

ANSWERS TO EXERCISES

Exercise 1:

i) $-3 < x \leq 4$

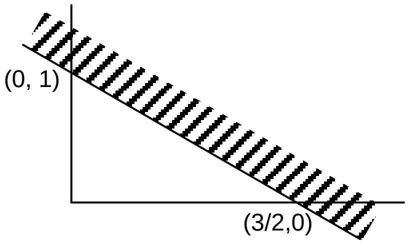
ii) $4 < x < 9$

iii) $2 \leq x < 8$

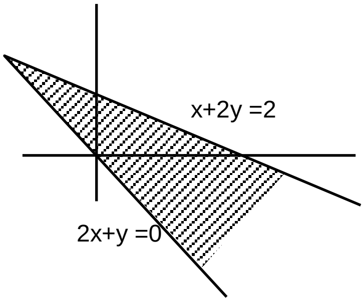
iv) $x > 4$

Exercise 2:

i)

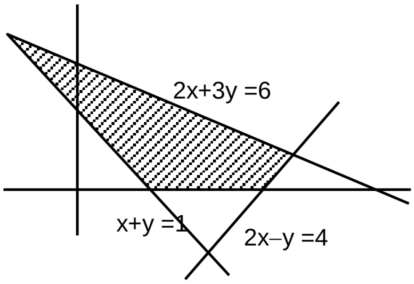


ii)



(iii) Empty set or null set.

iv)



SOLVED PROBLEMS

Problem 1: Solve for x, the following system of inequations:
$x + 2y \le 3$, $3x + 4y \ge 12$, $x \ge 0$, $y \ge 1$.

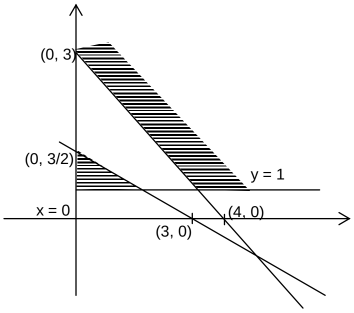
Solution: The solution region is bounded by the straight lines

$x + 2y = 3$, ... (1)

$3x + 4y = 12$, ... (2)

$x = 0$, ... (3)

$y = 1$. ... (4)



The straight lines (1) and (2) meet the x-axis in (3, 0) and (4, 0) and for (0, 0),
$x + 2y \le 3 \Rightarrow 0 \le 3$, which is true.
Hence (0, 0) lies in the half plane $x + 2y \le 3$. Also the lines (1) and (2) meet the y-axis in
(0, 3/2) and (0, 3) and for (0, 0)
$3x + 4y \ge 12$
$\Rightarrow 0 \ge 12$ which is not true. Hence (0, 0) doesn't belong to the half plane
$3x + 4y \ge 12$. Also $x \ge 0$, $y \ge 1 \Rightarrow$ that the solution set belongs to the first quadrant. Moreover all the boundary lines are part of the solution. From the shaded region, we find that there is no solution of the given system. Hence the solution set is an empty set.

Problem 2. Find all the pairs of consecutive odd natural numbers, both of which are larger than 10, such that their sum is less than 40.

Solution: Let x be one of the odd natural numbers so that the other one is x + 2 (being consecutive odd numbers). Here $x + 2 > x \Rightarrow x > 10$.
Also $x + x + 2 < 40 \Rightarrow 2x < 38$ or $x < 19$.
Hence $10 < x < 19$, so that the required pairs are (11, 13), (13, 15), (15, 17), (17, 19).

Problem 3. Solve $|3x - 2| \le \frac{1}{2}$.

Solution: Since $|3x - 2| \le \frac{1}{2}$, we have

$$-\frac{1}{2} \le 3x - 2 \le \frac{1}{2} \text{ or, } -\frac{1}{2} + 2 \le 3x \le \frac{1}{2} + 2$$

$$\text{or, } \frac{1}{3}\cdot\frac{3}{2} \le x \le \frac{1}{3}\cdot\frac{5}{2} \text{ or, } \frac{1}{2} \le x \le \frac{5}{6}.$$

Hence solution set consists of all real numbers lying between
$\frac{1}{2}$ and $\frac{5}{6}$ including $\frac{1}{2}$ and $\frac{5}{6}$.

Problem 4. A man wants to cut three lengths from a single piece of wire of total length 91 cm. The second length is to be 3 cm longer than the shortest and the third length is to be twice as long as the shortest. What are the possible lengths of the shortest piece if the third length is to be at least 5 cm longer than the second.

Solution: Let the length of the shortest piece of wire be x so that the second and the third lengths are x + 3 and 2x respectively. Now, it is given that third length is to be at least 5 cm longer than the second $\Rightarrow$ $2x \geq (x+3)+5$

or $x \geq 8$. (1)

Also the total length is 91 cm so that

$x + (x + 3) + (2x) \leq 91 \Rightarrow 4x \leq 88$

or, $x \leq 22$. (2)

From (1) and (2), we find that $8 \leq x \leq 22$.

Problem 5. Solve, for x, the inequations (i) $\frac{x+8}{x+2} > 1$, (ii) $\frac{x-5}{x+2} < 0$.

Solution: (i) The given inequation is rearranged as $\frac{x+8}{x+2} - 1 > 0$ or $\frac{6}{x+2} > 0$

$\Rightarrow x + 2 > 0$ or, $x > -2$.

(ii) For $\frac{x-5}{x+2} < 0$, we first consider that x + 2 > 0 so that x – 5 < 0 or x < 5. Combining the two results, we get that –2 < x < 5.

If, on the other hand, x + 2 < 0, then x – 5 > 0

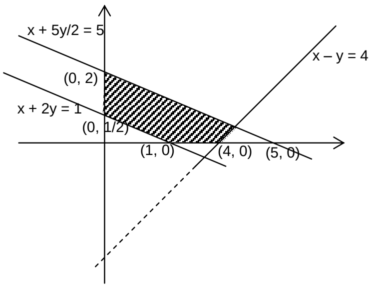
$\Rightarrow$ x < –2 and x > 5 which is not possible.

Hence the only solution is –2 < x < 5.

Problem 6. Draw the graph for the solution region satisfied by the inequalities

$$x + \frac{5}{2}y \leq 5,\ \ x + 2y \geq 1,\ \ x - y \leq 4,\ \ x \geq 0,\ \ y \geq 0.$$

Solution: The solution region lies in the first quadrant and is bounded by the straight lines



$x + \frac{5}{2}y = 5$, (1)

$x + 2y = 1$, (2)

$x - y = 4$. (3)

The line (1) meets the coordinate axes at (5, 0) and (0, 2). For (0, 0), $x + \frac{5}{2}y \leq 5 \Rightarrow 0 \leq 5$ which is true.

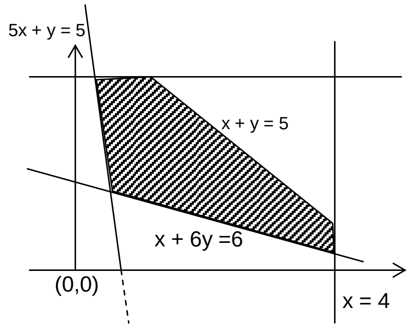
The line (2) meets the coordinate axes at (1, 0) and (0, 1/2). For (0, 0), $0 \geq 1$ which is not true. Also the line (3) meets the coordinate axes at (4, 0) and (0, –4).

For (0, 0), $0 \leq 4$ which is true.

The solution set belongs to the shaded region, which is an enclosed region bounded by the straight lines (1), (2), (3) and x = 0, y = 0.

Problem 7. Find the solution region satisfied by the inequalities
$x + y \le 5$, $x \le 4$, $y \le 4$, $x \ge 0$, $y \ge 0$, $5x + y \ge 5$, $x + 6y \ge 6$.

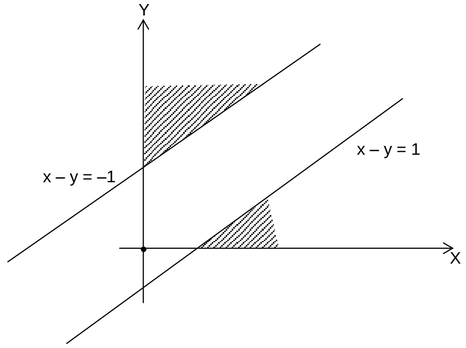
Solution: We find that the solution set satisfies $x \ge 0$, $y \ge 0$, $x \le 4$, $y \le 4$ so that the solution region lies within the square enclosed by the lines $x = 0$, $y = 0$, $x = 4$,
$y = 4$. Moreover, the solution region is bounded by the lines



$x + y = 5$,... (1)
$5x + y = 5$, ... (2)
$x + 6y = 6$. ... (3)
Line (1) meets the coordinate axes in (5, 0) and (0, 5) and the lines $x = 4$ and $y = 4$ in (4, 1) and (1, 4), and $0 < 5$ is true.
Hence (0, 0) belongs to the half plane $x + y \le 5$. But (0, 0) does not belong to the half planes $5x + y \ge 5$ and $x + 6y \ge 6$. The line $5x + y = 5$, meets the coordinate axes in (1, 0) and (0, 5), and meets the line $x = 4$ in (4, 1), where as it meet the line $y = 4$ in (1/5, 4).
Similarly $x + 6y = 6$ meets $x = 4$ in (4, 1/3) and $y = 4$ in (–18, 4).
The solution is marked as the shaded region.

Problem 8. Draw the graph of the solution region of the inequations $x \ge 0$, $y \ge 0$, $x - y \ge 1$,
$x - y \le -1$.

Solution: Here the region belongs to the first quadrant as $x \ge 0$, $y \ge 0$.
Moreover, the region is bounded by the straight lines



$x - y = 1$, (1)
$x - y = -1$. (2)
These are parallel lines.
Moreover, both $0 \ge 1$ and $0 \le -1$ are not true. Hence the origin does not belong to the half planes $x - y \ge 1$ and
$x - y \le -1$
$\Rightarrow$ Both the half planes are away from the origin and are disjointed.
Hence the solution set is empty.

Problem 9. Solve the system: $|2x - 3| \le 11$, $|x - 2| \ge 3$.

Solution: $|2x - 3| \le 11 \Rightarrow -11 \le 2x - 3 \le 11$
or $-8 \le 2x \le 14 \Rightarrow -4 \le x \le 7$ ... (1)
Further, $|x - 2| \ge 3 \Rightarrow x - 2 \ge 3$ for $x > 2$
and $2 - x \ge 3$ for $x < 2$
$\Rightarrow x \ge 5$ or $x \le -1$. ... (2)

From (1) and (2), we find that $x \in [-4, -1] \cup [5, 7]$.

Problem 10. Solve for x: $\frac{2-3x}{5} < \frac{1-x}{3} < \frac{3+4x}{2}$.

Solution: The given system may be written as

$$\frac{2-3x}{5} < \frac{1-x}{3} \text{ and } \frac{1-x}{3} < \frac{3+4x}{2}$$

or $6 - 9x < 5 - 5x$ and $2 - 2x < 9 + 12x$

or $1 < 4x$ and $-7 < 14x$

or $x > \frac{1}{4}$ and $x > -\frac{1}{2}$.

Hence $x \in \left(\frac{1}{4}, \infty\right)$.

ASSIGNMENT PROBLEMS

1. Solve the system of inequations:
   (i) $2x - 7 < 11$, $3x + 4 < -5$,
   (ii) $4x - 5 < 11$, $-3x - 4 \geq 8$.

2. Represent the following inequations graphically in the two dimensional plane:
   (i) $x - 2y + 4 \leq 0$,
   (ii) $y + 8 \geq 2x$,
   (iii) $2x \leq 6 - 3y$.

3. Represent the following systems of inequations graphically:
   (i) $x + y \leq 9$, $y > x$, $x \geq 1$.
   (ii) $3x + 4y \leq 60$, $x + 3y \leq 30$, $x \geq 0$, $y \geq 0$.
   (iii) $2x + y \geq 4$, $x + y \leq 3$, $2x - 3y \leq 6$.
   (iv) $3x + 2y \geq 24$, $3x + y \leq 15$, $x \geq 4$.
   (v) $3x + 2y \leq 24$, $x + 2y \leq 16$, $x + y \leq 10$, $x \geq 0$, $y \geq 0$.

4. Find all pairs of consecutive even positive integers, both of which are larger than 5 such that their sum is less than 23.

5. In a drilling process, the temperature T (in degree Celsius) x km below the surface of the earth, was found to be given by $T = 30 + 25(x - 3)$, $3 < x < 15$.
   At what depth will the temperature be between $200°C$ and $300°C$.

6. An aeroplane can carry a maximum of 360 passengers. The airlines reserves at least 20 seats for business class. But at least 8 times as many passengers prefer to travel by economy class than by business class. Find the possible graphic distribution of passengers in the two classes.

7. A company makes chairs and tables. The company has two different types of machines on which the chairs and tables are manufactured. The time required in hours for manufacturing each item and total available time in a week is given below:

	Chairs	Tables	Total available time (in hours)
Machine 1	5	2	24
Machine 2	4	8	64

   Show graphically the sets of chairs and tables can possibly be manufactured in a week.

8. Solve the systems:
   (i) $2x - 3 \leq 5, \dfrac{2x+5}{x+7} > 3$ $\qquad \left(\text{Hint: Write } \dfrac{2x+5}{x+7} - 3 > 0\right)$
   (ii) $\dfrac{x-2}{x+2} \geq 3,\ 2x - 7 \leq 5$.

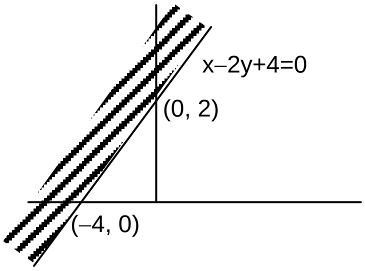
9. Solve for x : $\frac{x-3}{4} < \frac{2x-5}{5} < \frac{3x-5}{7}$.

10. A plumber can be paid either (i) Rs. 600 and Rs. 50 per hour or (ii) Rs. 170 per hour. If the job takes n hour, for what value of n the first mode i.e. (i) earns better wages for the plumber.

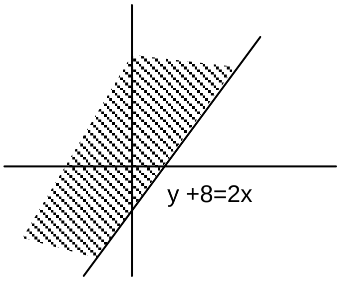
ANSWERS TO ASSIGNMENT PROBLEMS

1. (i) $x < -3$ (ii) $x \leq -4$

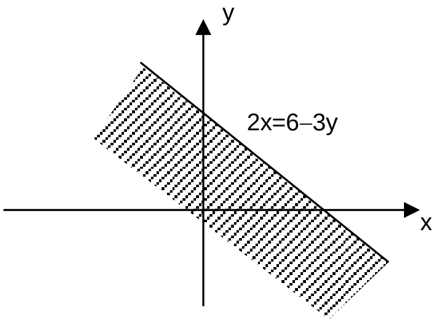
2. (i)



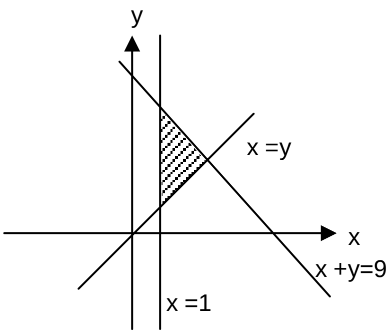
(ii)



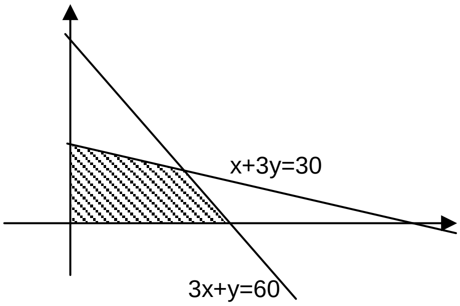
(iii)



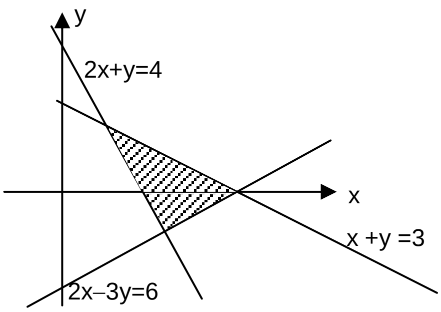
3. (i)



(ii)

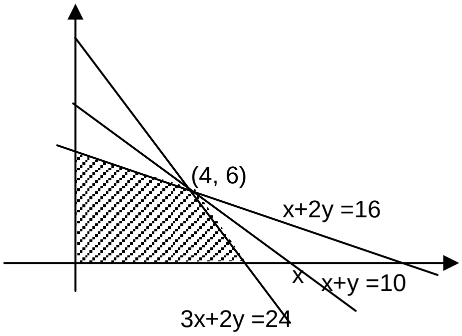


(iii)



(iv) null or empty set.

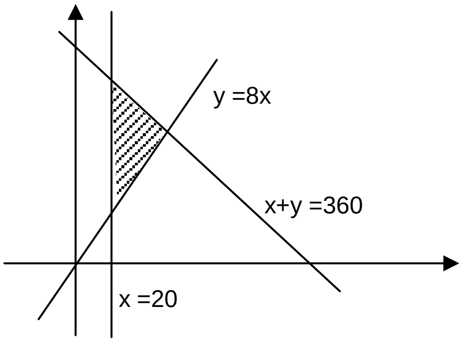
(v)



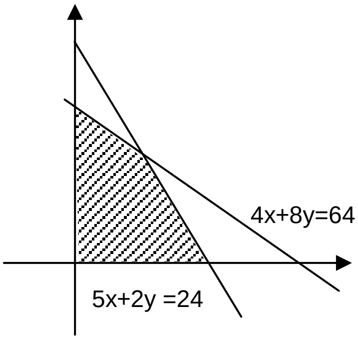
4. (6, 8), (8, 10), (10, 12).

5. Between 9.8m and 13.8m.

6. x = number in business class
y = number in economy class
$\Rightarrow x \geq 0, y \geq 0, x + y \leq 360, x \geq 20, y \geq x$.



7. x = number of chairs, y = number of tables
$\Rightarrow x \geq 0, y \geq 0, 5x + 2y \leq 24, 4x + 8y \leq 64$



8. (i) $x \in [-16, -7]$, (ii) $x \in [-4, -2)$

9. $x \in \left(\frac{1}{3}, \infty\right)$

10. $n < 5$